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SOLUTION OF THE DIFFERENTIAL EQUATION $dx^2 + dy^2 + dz^2 = ds^2$ AND ITS APPLICATION TO SOME GEOMETRICAL PROBLEMS.

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In this paper we give the solution of the equation

$$(1) \quad dx^2 + dy^2 + dz^2 = ds^2,$$

where x , y , and z are functions of a single parameter, in a different form from those given by E. Salkowski* and L. P. Eisenhart.† The new form shows that the coördinates of curves having the same differential element of arc differ from one another by the values of the coördinates of certain minimal curves. The formulæ for the direction cosines of the tangents, principal normals and binormals enable us to give simple solutions of some geometrical problems.

The equations of spherical curves are deduced, in which the differential of arc is expressed rationally.

1. **Parametric Representation of Curves.** If ψ and F are arbitrary analytic functions of a parameter u for a certain domain, an easy computation shows the truth of the following identity

$$(2) \quad \left(\frac{1-u^2}{2} \psi''' - uF'' \right)^2 + \left(i \frac{1+u^2}{2} \psi''' + iuF'' \right)^2 + (u\psi''' + F'')^2 = F''^2,$$

where the primes indicate the derivatives of the functions ψ and F with respect to u .

If we put

$$\begin{aligned} \frac{dx}{du} &= \frac{1-u^2}{2} \psi''' - uF'', & \frac{dy}{du} &= i \frac{1+u^2}{2} \psi''' + iuF'', \\ \frac{dz}{du} &= u\psi''' + F'', & \frac{ds}{du} &= F'', \end{aligned}$$

* E. Salkowski, Ueber algebraisch rectifizierbare Raumkurven, Math. Annalen, Vol. 67.

† L. P. Eisenhart, Fundamental parametric representation of space curves, Annals of Mathematics, Vol. 13.

and integrate, we get

$$(3) \quad \begin{aligned} x &= \frac{1-u^2}{2} \psi'' + u\psi' - \psi - uF' + F, \\ y &= i \left(\frac{1+u^2}{2} \psi'' - u\psi' + \psi + uF' - F \right), \\ z &= u\psi'' - \psi' + F', \quad s = F'. \end{aligned}$$

These expressions on account of (2) give the solution of the equation (1). Now x, y, z are the cartesian coördinates of a curve c and s is its arc.

The above solutions can be put in a different form by adding and subtracting

$$\frac{1-u^2}{2} F'', \quad i \frac{1+u^2}{2} F'', \quad \text{and} \quad uF''$$

to x, y, z respectively. Then if we put $\psi - F = \varphi$, the formulæ (3) become

$$(4) \quad \begin{aligned} x &= \frac{1-u^2}{2} \varphi'' + u\varphi' - \varphi + \frac{1-u^2}{2} F'', \\ y &= i \left(\frac{1+u^2}{2} \varphi'' - u\varphi' + \varphi + \frac{1+u^2}{2} F'' \right), \\ z &= u\varphi'' - \varphi' + uF'', \quad s = F'. \end{aligned}$$

The expressions (4) may be written

$$x = x_m + \xi, \quad y = y_m + \eta, \quad z = z_m + \zeta,$$

where x_m, y_m, z_m are the coördinates of the points of a minimal curve Γ , since

$$x_m = \frac{1-u^2}{2} \varphi'' + u\varphi' - \varphi, \quad y_m = i \left(\frac{1+u^2}{2} \varphi'' - u\varphi' + \varphi \right), \quad z_m = u\varphi'' - \varphi',$$

and ξ, η, ζ , containing F'' , namely

$$\xi = \frac{1-u^2}{2} F'', \quad \eta = i \frac{1+u^2}{2} F'', \quad \zeta = uF'',$$

are the coördinates of the points of a curve lying on a sphere of zero radius for $\xi^2 + \eta^2 + \zeta^2 = 0$.

Since $\frac{dx_m}{du}, \frac{dy_m}{du}, \frac{dz_m}{du}$ are proportional to ξ, η, ζ respectively, the points of C lie on the corresponding tangents to Γ .

2. **Generality of Equations (3).** The coördinates of any curve may be

written in the form (4). Suppose that a curve is given by the following equations

$$x = f_1(v), \quad y = f_2(v), \quad z = f_3(v),$$

where f_1, f_2, f_3 are analytic functions of v in a certain domain. Form

$$ds = [(f_1'(v))^2 + (f_2'(v))^2 + (f_3'(v))^2]^{1/2} dv.$$

From (4) we have

$$\begin{aligned} dx &= \left(\frac{1-u^2}{2} \varphi''' + \frac{1-u^2}{2} F''' - uF'' \right) du, \\ (5) \quad dy &= i \left(\frac{1+u^2}{2} \varphi''' + \frac{1+u^2}{2} F''' + uF'' \right) du, \\ dz &= (u\varphi''' + uF''' + F'') du, \quad ds = F'' du. \end{aligned}$$

Hence we have

$$dx - idy = (\varphi''' + F''') du, \quad dz - ds = u(\varphi''' + F''') du.$$

Therefore

$$(6) \quad u = \frac{dz - ds}{dx - idy}.$$

If the curve to be dealt with is a straight line then $dx/ds, dy/ds, dz/ds$ are constants and therefore u is a constant. Hence a straight line cannot be represented by the formulæ (4). To discuss fully the case of u a constant we consider another parameter

$$\bar{u} = \frac{dz + ds}{dx - idy},$$

which gives a similar representation of the coördinates of a curve as given by (4) in terms of the parameter u . In fact, by making

$$dx - idy = \psi''' du, \quad ds - dz = -2F'' - \bar{u}\psi''' du,$$

we get by the definition of \bar{u} the formulæ (3) and then pass to (4) in the indicated manner. We see now that in the case of a straight line both u and \bar{u} are constants and conversely, if u and \bar{u} are constants the curve under consideration is a straight line. But if u is a constant, \bar{u} may be a variable and conversely. For the relation between u and \bar{u} can be given either of the forms

$$u + \frac{2F''}{\varphi''' + F'''} = \bar{u}, \quad u = \bar{u} + \frac{2\bar{F}''}{\varphi''' + F'''}.$$

Suppose now u is a constant c , then

$$\bar{u} = c - \frac{2\bar{F}''}{\bar{\varphi}''' + \bar{F}'''}$$

and $(c - \bar{u})(\bar{\varphi}''' + \bar{F}''') = 2\bar{F}''$. Substituting into the analogues of (5) for \bar{u} , we get

$$\begin{aligned} dx &= (\bar{\varphi}''' + \bar{F}''') \frac{(1 + c\bar{u})}{2} d\bar{u}, & dy &= i(\bar{\varphi}''' + \bar{F}''') \frac{(1 - c\bar{u})}{2} d\bar{u}, \\ dz &= (\bar{\varphi}''' + \bar{F}''') \frac{(c + \bar{u})}{2} d\bar{u}, \end{aligned}$$

and hence

$$\frac{1 - c^2}{2} dx + i \frac{1 + c^2}{2} dy + cdz = 0,$$

or

$$(7) \quad \frac{1 - c^2}{2} x + i \frac{1 + c^2}{2} y + cz = A,$$

where A is an arbitrary constant. The equation (7) shows that the curve (x, y, z) lies in an isotropic plane in case either u or \bar{u} is a constant, and can be represented by equations of the form (4) either in terms of u or \bar{u} . So that only in the case of a straight line the representation (4) fails. From (4) we see that

$$(8) \quad \frac{1 - u^2}{2} x + \frac{i(1 + u^2)}{2} y + uz = -\varphi.$$

So that $\varphi, \varphi', \varphi''$ and F'' are now known and we can write the expressions for x, y, z in the form (4) except in the case of a straight line.

3. Algebraic Curves. If the curve under consideration is an algebraic curve we have

$$x = F_1(z), \quad y = F_2(z)$$

where F_1 and F_2 represent algebraic functions of z . Hence $dx/dz, dy/dz, ds/dz$ are algebraic functions of z , and from (6) z is an algebraic function of u . That is x, y, z are algebraic functions of u and from (8) φ is an algebraic function of u . It follows then that in the case of an algebraic curve F'' and φ must be algebraic functions of u . The converse is also true. For if F'' and φ are algebraic functions of u , x and y are algebraic functions of z and the curve is an algebraic curve. Therefore a necessary and sufficient condition that the curves represented in the form (4) be algebraic curves is that F'' and φ be algebraic functions of u . Since $ds = F'' du$, we see that if F' is an algebraic function of u , the algebraic curves corresponding to that value of s are algebraically rectifiable.

4. The Fundamental Formulæ for the Curves Represented by (4). We

denote the direction-cosines of the tangent, principal normal and binormal to a curve by α, β, γ ; l, m, n ; λ, μ, ν respectively. Then

$$\alpha = \frac{1-u^2}{2} \chi - u, \quad \beta = i \frac{1+u^2}{2} \chi + iu, \quad \gamma = u\chi + 1,$$

where

$$\chi = \frac{\varphi''' + F'''}{F''}.$$

Let us call σ the arc of the spherical indicatrix of the tangents, and σ_1 that of the binormals. Then $d\sigma = \sqrt{\chi^2 - 2\chi'} du$, and if ρ denotes the radius of curvature, we have

$$\rho = \frac{ds}{d\sigma} = \frac{F''}{\sqrt{\chi^2 - 2\chi'}}.$$

If we make use of Frenet's formulæ, we find

$$l = \rho \frac{d\alpha}{ds}, \quad m = \rho \frac{d\beta}{ds}, \quad n = \rho \frac{d\gamma}{ds},$$

so that the direction-cosines of the principal normals become

$$l = \frac{1}{\sqrt{\chi^2 - 2\chi'}} \left\{ \frac{1-u^2}{2} \chi' - u\chi - 1 \right\},$$

$$m = \frac{i}{\sqrt{\chi^2 - 2\chi'}} \left\{ \frac{1+u^2}{2} \chi' + u\chi + 1 \right\},$$

$$n = \frac{1}{\sqrt{\chi^2 - 2\chi'}} \{u\chi' + \chi\}.$$

For $\lambda = \beta n - \gamma m$, etc., we obtain

$$\lambda = \frac{-i}{\sqrt{\chi^2 - 2\chi'}} \left\{ \frac{1-u^2}{2} (\chi' - \chi^2) + u\chi + 1 \right\},$$

$$\mu = \frac{1}{\sqrt{\chi^2 - 2\chi'}} \left\{ \frac{1+u^2}{2} (\chi' - \chi^2) - u\chi - 1 \right\},$$

$$\nu = \frac{-i}{\sqrt{\chi^2 - 2\chi'}} \{u(\chi' - \chi^2) - \chi\},$$

and

$$d\sigma_1 = i \cdot \frac{\chi^3 - 3\chi\chi' + \chi''}{\chi^2 - 2\chi'} du.$$

If τ denotes the radius of torsion, then

$$\tau = \frac{ds}{d\sigma_1} = -i \frac{(\chi^2 - 2\chi')F''}{\chi^3 - 3\chi\chi' + \chi''}, \quad \frac{\rho}{\tau} = i \frac{\chi^3 - 3\chi\chi' + \chi''}{(\chi^2 - 2\chi')^{3/2}}.$$

Since

$$d\sigma_1 = i \left(\chi - \frac{\chi\chi' - \chi''}{\chi^2 - 2\chi'} \right) du$$

we have

$$\sigma_1 = i \left(\int \chi du - \log \sqrt{\chi^2 - 2\chi'} \right).$$

Therefore if χ is the derivative of a function $M(u)$, σ_1 is expressible without any sign of integration. We shall see that this is the case for all spherical curves.

5. Plane Curves. By using formulæ (3) we obtain the coördinates of a plane curve in terms of the parameter u . For the sake of simplicity we suppose that the plane of the curve is the xy -plane. Hence, if we take $z = 0$, we find that to within an additive constant

$$F = -u\psi' + 2\psi.$$

Substituting the values of F and F' in (3), we get

$$x = \frac{1+u^2}{2} \psi'' - u\psi' + \psi, \quad y = i \left(\frac{1-u^2}{2} \psi'' + u\psi' - \psi \right),$$

$$s = F' = \psi' - u\psi''.$$

The formulæ for α , β , γ are quite interesting for the plane curves in terms of the parameter u . They are

$$\alpha = -\frac{1+u^2}{2u}, \quad \beta = -i\frac{1-u^2}{2u}, \quad \gamma = 0.$$

Since $dz/ds = u\chi + 1$, in the case of plane curves $\chi = -1/u$. So that for all plane curves the ratio $(\varphi''' + F''')/F''$ has the same value, $-1/u$. Another interesting fact is that the slopes of the tangents to the plane curves are given by the formula

$$i\frac{1-u^2}{1+u^2},$$

and are independent of the function ψ . We find also for the plane curves

$$\rho = iu^2\psi''',$$

and for the tangent lines the equation

$$i(1+u^2)y + (1-u^2)x + 2(u\psi' - \psi) = 0$$

6. Spherical Curves. To deduce the equations for spherical curves we assume that the center of the sphere of radius unity is at the origin so that the coördinates x , y , z of the spherical curve satisfy the equation

$$x^2 + y^2 + z^2 = 1,$$

which for the expressions (4) necessitates the relation

$$\varphi'^2 - 2\varphi\varphi'' - 2\varphi F'' = 1.$$

Hence

$$F'' = \frac{\varphi'^2 - 1 - 2\varphi\varphi''}{2\varphi},$$

and the equations (4) for a spherical curve become

$$\begin{aligned} x &= \frac{1-u^2}{2} \varphi'' + u\varphi' - \varphi + \frac{1-u^2}{2} \cdot \frac{\varphi'^2 - 2\varphi\varphi'' - 1}{2\varphi}, \\ y &= i \left(\frac{1+u^2}{2} \varphi'' - u\varphi' + \varphi \right) + i \frac{1+u^2}{2} \cdot \frac{\varphi'^2 - 2\varphi\varphi'' - 1}{2\varphi}, \\ z &= u\varphi'' - \varphi' + u \frac{\varphi'^2 - 2\varphi\varphi'' - 1}{2\varphi}, \\ ds &= \frac{\varphi'^2 - 2\varphi\varphi'' - 1}{2\varphi} du. \end{aligned}$$

The function designated by χ we obtain by computing $(\varphi''' + F''')/F''$ and find

$$\chi = \frac{-\varphi'}{\varphi} = -\frac{d \log \varphi}{du}.$$

Substituting this value of χ into the formulæ of § 4 we obtain the corresponding formulæ for spherical curves. We call attention to the formula for $d\sigma_1$, which after integration is reduced to

$$\sigma_1 = i \log \frac{1}{\varphi} \cdot \frac{d\sigma}{du}.$$